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On the pure spinor heterotic superstring b ghost

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ABSTRACT: A simplified pure spinor superstring b ghost in a curved heterotic background was constructed recently. The b ghost is a composite operator and it is not holomorphic. However, it satisfies $\bar{\partial}b = [Q, \Omega]$, where Q is the BRST charge. In this paper, we find a possible Ω .

KEYWORDS: Superspaces, Superstrings and Heterotic Strings

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1 Introduction

The minimal pure spinor superstring action in a generic supergravity and super-Yang-Mills (SYM) heterotic background first appeared in [1]. It was also shown in [1] that imposing that both the BRST operator is nilpotent and the BRST current is holomorphic all the supergravity and super-Yang-Mills superfield constraints are reproduced. However, the b ghost in a curved heterotic background was only fully constructed recently [2].

In the pure spinor formalism, the b ghost is a composite operator and it is constructed as a solution to the equation $\delta_B b = T$, where δ_B means BRST variation and T is the energy-momentum tensor. In order to construct a covariant b ghost in a flat space background, it is necessary to use the non-minimal pure spinor formalism introduced in [3]. The left-moving ghost sector of the minimal formalism consist of a pure spinor λ^α and its conjugate momentum ω_α . The non-minimal formalism has the following additional variables: a bosonic pure spinor $\hat{\lambda}_\alpha$ and a constrained fermion r_α together with their conjugate momenta $\hat{\omega}^\alpha$ and s^α respectively. In [4], the b ghost in a flat space background was simplified, it was rewritten in terms of RNS-like variables.

The BRST transformations of all the minimal fields in a curved heterotic background were computed in [5] and they were used to show that the heterotic minimal action is BRST invariant. The non-minimal formalism in a curved heterotic background was constructed in [6]. The same set of non-minimal variables described previously for the flat space background was added to the model and their BRST transformations were found requiring that the BRST transformations of the minimal variables are unchanged and imposing some consistency conditions. This non-minimal heterotic formalism was used to construct the heterotic b ghost. A simplified version of the curved heterotic b ghost, and the one used in this work, was constructed in [2].

In a flat space background b is a holomorphic operator, i.e. $\bar{\partial}b = 0$ where $\bar{\partial}$ is the antiholomorphic derivative. However, it is not holomorphic in a generic curved background and it is expected to satisfy $\bar{\partial}b = \delta_B \Omega$ for some Ω . Note that Ω is defined up to a BRST exact term. In fact, Ω has been constructed in a Type IIB $AdS_5 \times S^5$ background in [7] and in a super-Maxwell background in [8]. In this work, we find a possible Ω for the curved heterotic b ghost of [2].

The pure spinor b ghost in a generic curved Type II background has not been constructed yet. It is only known for particular R-R backgrounds [7], being $AdS_5 \times S^5$ an example. Recently, the non-minimal pure spinor formalism in a generic curved Type II background was constructed in [9]. It is expected that the b ghost can be constructed using this new non-minimal formalism.

This paper is organized as follows. In section 2, we review both the minimal and non-minimal pure spinor formalisms in a curved heterotic background. Moreover, the expression of the b ghost is presented and all the necessary equations of motion to compute $\bar{\partial}b$ are derived. In section 3, we explicitly construct Ω firstly in a super-Yang-Mills heterotic background, then in the case $s^\alpha = \hat{\omega}^\alpha = 0$ and finally the general case is considered. The complete expression for Ω is given in section 4. The appendix has all the conventions used in this work and some important identities.

2 Review of the pure spinor formalism in a curved heterotic background

In this section, we review both the minimal and non-minimal pure spinor heterotic superstring. Moreover, the expression of the b ghost is presented. We also derive all the necessary equations of motion to evaluate $\bar{\partial}b$. All the conventions used can be found in the appendix.

2.1 The minimal pure spinor formalism

The minimal pure spinor heterotic superstring action S_m is written in terms of the $\mathcal{N} = 1$ $D = 10$ superspace coordinates and background superfields. We will denote the curved superspace coordinates by Z^M with $M = (m, \mu)$ and $m = 0, \dots, 9$, $\mu = 1, \dots, 16$. The tangent superspace indices will be denoted by $A = (a, \alpha)$, with $a = 0, \dots, 9$, and $\alpha =$

1, ..., 16. The action in a generic curved heterotic background is [1]

$$S_m = \int d^2z \left[\frac{1}{2} \Pi^a \bar{\Pi}^b \eta_{ab} + \frac{1}{2} \Pi^A \bar{\Pi}^B B_{BA} + d_\alpha \bar{\Pi}^\alpha + w_\alpha \bar{\nabla} \lambda^\alpha \right. \\ \left. + J_I \left(\Pi^A A_A^I + d_\alpha W^{I\alpha} + \frac{1}{2} \lambda^\alpha w_\beta U_\alpha^{I\beta} \right) + S_J + \bar{b} \partial \bar{c} \right], \quad (2.1)$$

with

$$\Pi^A = \partial Z^M E_M^A, \quad \bar{\Pi}^A = \bar{\partial} Z^M E_M^A,$$

where E_M^A is the supervielbein matrix. In the action, J_I are right-moving $E_8 \times E_8$ or $SO(32)$ currents with $I = 1, \dots, 496$ and their action is S_J , whose explicit form will not be needed. The (\bar{b}, \bar{c}) are the usual right-moving Virasoro ghosts of the bosonic string. The additional ghosts are $(\lambda^\alpha, \omega_\alpha)$ with ω_α being the conjugate momentum of λ^α and λ^α a bosonic pure spinor, which means that it satisfies

$$\lambda \gamma_a \lambda = 0,$$

where $(\gamma_a)_{\alpha\beta}$ are tangent $SO(1,9)$ generalized Pauli matrices. The covariant derivative that appears in the action is given by

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + \lambda^\beta \bar{\Pi}^A \Omega_{A\beta}{}^\alpha,$$

with $\Omega_{M\beta}{}^\alpha$ the background spin connection. The additional background superfields appearing in the action are: the two-form potential B_{MN} , the super-Yang-Mills potential A_M^I and finally, the super-Yang-Mills field-strengths $W^{I\alpha}$ and $U_\alpha^{I\beta}$.

Due to the fact that λ^α is a constrained variable, its momentum conjugate ω_α is defined up to the gauge transformation

$$\delta \omega_\alpha = \Lambda^a (\gamma_a \lambda)_\alpha, \quad (2.2)$$

for any Λ^a . Imposing that the action is gauge invariant implies that the superfields $\Omega_{M\alpha}{}^\beta$ and $U_\alpha^{I\beta}$ have the decompositions

$$\Omega_{A\alpha}{}^\beta = \Omega_A \delta_\alpha^\beta + \frac{1}{4} \Omega_{Aab} (\gamma^{ab})_\alpha{}^\beta, \quad U_\alpha^{I\beta} = U^I \delta_\alpha^\beta + \frac{1}{4} U_{ab}^I (\gamma^{ab})_\alpha{}^\beta, \quad (2.3)$$

which is equivalent to the fact that ω_α can only appear in the gauge invariant combinations $J = \lambda^\alpha \omega_\alpha$ and $N^{ab} = \frac{1}{2} (\lambda \gamma^{ab} \omega)$. In the decomposition of the spin connection above, Ω_A is the scaling connection and Ω_{Aab} is the usual Lorentz connection satisfying $\Omega_{Mab} = -\Omega_{Mba}$.

The action S_m is BRST invariant and the left-moving BRST operator is

$$Q = \oint dz \lambda^\alpha d_\alpha. \quad (2.4)$$

Note that the variable d_α also appears in the action. The pure spinor BRST operator was first constructed in a flat space background in [10] and its origin, at least for the flat

space case, was only discovered recently as a result of the procedure of gauge-fixing a more fundamental action [11, 12].

The pure spinor formalism is well defined only if the BRST operator is nilpotent and the BRST current is holomorphic, which means $\bar{\partial}(\lambda d) = 0$. In [1] the constraints on the background superfields coming from these two conditions were found and solved and it was shown that they are the correct $\mathcal{N} = 1$ supergravity and super-Yang-Mills constraints after fixing all the gauge symmetries. In this work, we will need the following set of these constraints

$$\begin{aligned} T_{A\beta}{}^{\alpha} &= H_{ba\beta} = H_{\alpha\beta\gamma} = T_{\beta(ab)} = 0, & H_{\alpha\beta b} &= T_{\alpha\beta b}, & (2.5) \\ \lambda^{\beta}\lambda^{\gamma}R_{b\beta\gamma}{}^{\alpha} &= 0, & F_{\alpha\beta}^I &= 0, & F_{a\beta}^I &= W^{I\alpha}(\gamma_a)_{\alpha\beta}, & \frac{1}{2}U_{\beta}^{I\alpha} &= \nabla_{\beta}W^{I\alpha}, \\ T_{\alpha\beta}{}^a &= \gamma_{\alpha\beta}^a, & T_{\alpha a}{}^b &= 2(\gamma_a^b)_{\alpha}{}^{\beta}\Omega_{\beta}, \end{aligned}$$

where H are the three-form field-strengths, T are the torsions, R are the curvatures, F are the SYM field-strengths and ∇_M are covariant derivatives defined in terms of the spin connections and SYM connections, see the appendix for precise definitions.

In fact, the action S_m of (2.1) is not the complete action. The Fradkin-Tseytlin term that couples the background dilaton superfield Φ with the worldsheet curvature is absent. This term was not written down because it is higher order in α' and it is not needed for classical conformal invariance of the action. However, it is necessary for quantum conformal invariance [1, 13] and homomorphicity of the BRST current (λd) , which imply the additional constraint

$$\Omega_{\alpha} = \frac{1}{4}\nabla_{\alpha}\Phi. \quad (2.6)$$

Combining the set of constraints given above with the Bianchi identities, one can derive additional relations among the background superfields. Using that $\nabla_A\gamma_{\alpha\beta}^b = -2\Omega_A\gamma_{\alpha\beta}^b$ and the Bianchi identity involving the super-Yang-Mills field-strength F_{AB}^I given in (A.3), one has

$$\nabla_{[a}F_{\alpha\beta]}^I + T_{[a\alpha}{}^F F_{|F|\beta]}^I = 0 \Rightarrow \frac{1}{2}U^I + \Omega_{\alpha}W^{I\alpha} = 0 \text{ and } F_{ab}^I = \frac{1}{2}U_{ab}^I + T_{\alpha ab}W^{I\alpha}. \quad (2.7)$$

Moreover, three important properties of the background superfields can be derived from a combination of two Bianchi identities. Consider the Bianchi identities (A.2) and (A.5) with the following specification of the indices

$$\eta^{ab}(\nabla_{[a}T_{\alpha\beta]b} - R_{[a\alpha\beta]b} + T_{[a\alpha}{}^AT_{|A|\beta]b}) = 0, \text{ and } \nabla_{[a}H_{b\alpha\beta]} + \frac{3}{2}T_{[ab}{}^AH_{|A|\alpha\beta]} = 0,$$

one can show that the Bianchi identities above imply

$$\Omega_a = 0, \quad \eta^{ab}T_{cab} = 0, \quad T_{abc} + H_{abc} = 0. \quad (2.8)$$

In what follows, the explicit form of R_{AB} , which are the components of the curvature constructed from the scaling connection Ω_A , will be needed. Note that for generic values of A and B , the curvature has the form

$$R_{AB} = \nabla_{[A}\Omega_{B]} + T_{AB}{}^C\Omega_C,$$

and due to the constraints above, we have

$$R_{a\beta} = \nabla_a \Omega_\beta, \quad R_{ab} = T_{ab}{}^\alpha \Omega_\alpha, \quad R_{\alpha\beta} = \nabla_{(\alpha} \Omega_{\beta)} \Rightarrow R_{\alpha\beta} = -\frac{1}{4} \gamma_{\alpha\beta}^a \nabla_a \Phi. \quad (2.9)$$

2.1.1 The BRST transformations

As mentioned before, the minimal action S_m of (2.1) is BRST invariant. The BRST transformations of all the variables appearing on the minimal action were derived in [5] expressing d_α as a function of the other variables and the conjugate momenta and using the canonical commutation relations. The transformations of λ^α , ω_α , d_α and J_I are

$$\begin{aligned} \delta_B \lambda^\alpha &= -\lambda^\beta \Omega_{\beta\gamma}{}^\alpha \lambda^\gamma, & \delta_B \omega_\alpha &= \lambda^\beta \Omega_{\beta\alpha}{}^\gamma \omega_\gamma + d_\alpha, & \delta_B J_I &= J_K f_{JI}^K \lambda^\alpha A_\alpha^J, \\ \delta_B d_\alpha &= \lambda^\beta \Omega_{\beta\alpha}{}^\gamma d_\gamma + \lambda^\beta \lambda^\gamma \omega_\delta R_{\alpha\beta\gamma}{}^\delta + \Pi^a (\lambda \gamma_a)_\alpha. \end{aligned} \quad (2.10)$$

In addition, the BRST transformations of Π^A are

$$\delta_B \Pi^a = -\lambda^\alpha \Omega_{\alpha b}{}^a \Pi^b - \lambda^\alpha \Pi^A T_{A\alpha}{}^a, \quad \delta_B \Pi^\beta = \nabla \lambda^\beta - \lambda^\alpha \Omega_{\alpha\gamma}{}^\beta \Pi^\gamma. \quad (2.11)$$

The transformations of the variables $\bar{\Pi}^A$ can be obtained from the ones above by replacing both Π with $\bar{\Pi}$ and ∂ with $\bar{\partial}$. Finally, let Ψ be any superfield, it transforms as

$$\delta_B \Psi = \lambda^\alpha \partial_\alpha \Psi. \quad (2.12)$$

In the expression of the heterotic b ghost derived in [2] which will be reviewed in a future subsection, the following combination of variables and superfields appears

$$D_\alpha = d_\alpha + \frac{1}{4} \lambda^\beta T_{\beta ab} (\gamma^{ab})_\alpha{}^\gamma \omega_\gamma - 3(\lambda \Omega) \omega_\alpha. \quad (2.13)$$

The BRST transformation of D_α was derived in [2]. It was shown that using the constraints of the superfields and the Bianchi identities, the transformation simplifies to

$$\delta_B D_\alpha = \lambda^\beta \Omega_{\beta\alpha}{}^\gamma D_\gamma + 3(\lambda \Omega) D_\alpha + \Pi^a (\lambda \gamma_a)_\alpha - \frac{1}{4} \lambda^\beta T_{\beta ab} (\gamma^{ab})_\alpha{}^\gamma D_\gamma. \quad (2.14)$$

One last comment is that sometimes the computations are simplified once one writes the BRST transformation of ω_α in terms of D_α , it takes the form

$$\delta_B \omega_\alpha = \lambda^\beta \Omega_{\beta\alpha}{}^\gamma \omega_\gamma + D_\alpha + 3(\lambda \Omega) \omega_\alpha - \frac{1}{4} \lambda^\beta T_{\beta ab} (\gamma^{ab})_\alpha{}^\gamma \omega_\gamma. \quad (2.15)$$

2.2 The non-minimal pure spinor formalism

The non-minimal pure spinor formalism in a flat background was developed in [3]. One of the motivations for adding new variables to the minimal formalism was that using these new variables one can construct a covariant b ghost which is important for multi-loop calculations. The new variables are: a bosonic pure spinor $\hat{\lambda}_\alpha$ together with its conjugate momentum $\hat{\omega}^\alpha$ and a constrained fermionic spinor r_α and its conjugate momentum s^α . They satisfy

$$\hat{\lambda} \gamma_a \hat{\lambda} = 0, \quad \hat{\lambda} \gamma_a r = 0, \quad (2.16)$$

and due to the above constraints the conjugate momenta are defined up to the gauge transformations

$$\delta\hat{\omega}^\alpha = \hat{\Lambda}^a(\gamma_a\hat{\lambda})^\alpha - \phi^a(\gamma_a r)^\alpha, \quad \delta s^\alpha = \phi^a(\gamma_a\hat{\lambda})^\alpha,$$

where $\hat{\Lambda}^a$ and ϕ^a are arbitrary parameters.

The non-minimal pure spinor formalism in a curved heterotic background was constructed in [6]. Again, one of the motivations for extending the minimal formalism was the construction of the b ghost and the same set of new variables was added. The first step of the construction was finding the BRST transformations of the non-minimal variables after modifying the BRST operator by adding a new term equal to the flat space case. This was achieved by noticing that the transformations of the minimal fields do not change and imposing consistency of the BRST transformations. The answer is

$$\begin{aligned} \delta_B \hat{\lambda}_\alpha &= -r_\alpha + \lambda^\gamma \hat{\lambda}_\beta \left(\Omega_{\gamma\alpha}{}^\beta - \frac{1}{4} T_{\gamma ab} (\gamma^{ab})_\alpha{}^\beta \right), & \delta_B \hat{\omega}^\alpha &= -\hat{\omega}^\beta \lambda^\gamma \left(\Omega_{\gamma\beta}{}^\alpha - \frac{1}{4} T_{\gamma ab} (\gamma^{ab})_\beta{}^\alpha \right), \\ \delta_B s^\alpha &= \hat{\omega}^\alpha + s^\beta \lambda^\gamma \left(\Omega_{\gamma\beta}{}^\alpha - \frac{1}{4} T_{\gamma ab} (\gamma^{ab})_\beta{}^\alpha \right), & \delta_B r_\alpha &= -\lambda^\gamma r_\beta \left(\Omega_{\gamma\alpha}{}^\beta - \frac{1}{4} T_{\gamma ab} (\gamma^{ab})_\alpha{}^\beta \right). \end{aligned} \quad (2.17)$$

The non-minimal pure spinor action that we are going to use includes a torsion term as the one in [2]. The action is

$$S_{nm} = \delta_B \int d^2 z \left(s \bar{\nabla} \hat{\lambda} + \frac{1}{4} \bar{\Pi}^A T_{Aab} (s \gamma^{ab} \hat{\lambda}) \right),$$

with

$$\bar{\nabla} \hat{\lambda}_\alpha = \bar{\partial} \hat{\lambda}_\alpha - \hat{\lambda}_\beta \bar{\Pi}^A \Omega_{A\alpha}{}^\beta,$$

and after computing the BRST transformations, one has

$$\begin{aligned} S_{nm} &= \int d^2 z \left[\hat{\omega} \bar{\nabla} \hat{\lambda} + s \bar{\nabla} r + \frac{1}{4} \bar{\Pi}^A T_{Aab} [(\hat{\omega} \gamma^{ab} \hat{\lambda}) + (s \gamma^{ab} r)] \right. \\ &\quad \left. + \lambda^\alpha \bar{\Pi}^A R_{A\alpha} (s \hat{\lambda}) + \frac{1}{4} \lambda^\alpha \bar{\Pi}^c \mathcal{R}_{c\alpha ab} (s \gamma^{ab} \hat{\lambda}) \right], \end{aligned} \quad (2.18)$$

and we have used the definition

$$\mathcal{R}_{c\alpha ab} = R_{c\alpha ab} - \nabla_{[c} T_{\alpha]ab} - T_{c\alpha}{}^d T_{dab} + T_{cd[a} T_{b]\alpha}{}^d, \quad (2.19)$$

where $[]$ means antisymmetrization without any additional numerical factor. Note that replacing c by β in the above definition gives $\mathcal{R}_{\beta\alpha ab} = 0$ due to the Bianchi identity (A.2).

2.3 The curved heterotic b ghost

In the pure spinor formalism the b ghost is a composite operator and it is constructed as a solution to the equation $\delta_B b = T$ where T is the energy-momentum tensor. In a flat space background this equation was solved in [3] using the non-minimal formalism. The result

was rewritten using RNS-like vector variables in a simpler way in [4]. The b ghost in a curved heterotic background was first constructed in [6], however in this paper we will use the simplified version of the b ghost of [2] where generalized RNS-like variables were used. The b ghost is given by

$$b = -s^\alpha \nabla \hat{\lambda}_\alpha - \frac{1}{4} \Pi^A T_{Aab} (s \gamma^{ab} \hat{\lambda}) - \omega_\alpha \Pi^\alpha + \frac{1}{2(\lambda \hat{\lambda})} (\omega \gamma_a \hat{\lambda}) (\lambda \gamma^a \Pi) + \Pi^a \bar{F}_a - \frac{1}{4(\lambda \hat{\lambda})} (\lambda \gamma^{ab} r) \bar{F}_a \bar{F}_b, \quad (2.20)$$

where \bar{F}_a are the generalized RNS-like variables defined by

$$\bar{F}_a = -\frac{1}{2(\lambda \hat{\lambda})} (D \gamma_a \hat{\lambda}) - \frac{1}{8(\lambda \hat{\lambda})^2} (r \gamma_{abc} \hat{\lambda}) N^{bc}, \quad (2.21)$$

with D_α defined in (2.13) and $N^{bc} = \frac{1}{2} (\lambda \gamma^{bc} \omega)$.

2.4 The equations of motion

In this subsection, we compute all the necessary equations of motion for evaluating $\bar{\partial} b$. The complete curved heterotic action is the sum of the minimal action S_m of (2.1) and the non-minimal action S_{nm} of (2.18). Varying λ^α , ω_α and d_α , one obtains respectively

$$\begin{aligned} \bar{\partial} \omega_\alpha &= \bar{\Pi}^A \Omega_{A\alpha}{}^\beta \omega_\beta + \frac{1}{2} J_I U_\alpha^{I\beta} \omega_\beta + \bar{\Pi}^A R_{A\alpha} (s \hat{\lambda}) + \frac{1}{4} \bar{\Pi}^c \mathcal{R}_{c\alpha ab} (s \gamma^{ab} \hat{\lambda}), \\ \bar{\partial} \lambda^\alpha &= -\lambda^\beta \bar{\Pi}^A \Omega_{A\beta}{}^\alpha - \frac{1}{2} \lambda^\beta J_I U_\beta^{I\alpha}, \quad \bar{\Pi}^\alpha = -J_I W^{I\alpha}, \end{aligned} \quad (2.22)$$

where $\mathcal{R}_{c\alpha ab}$ was defined in (2.19).

One way of obtaining the equation of motion of J_I is writing S_J in (2.1) explicitly and varying the action with respect to the right-moving variables. Another way, which is independent of any specific representation of J_I , is to note that J_I transforms under gauge transformations and when $W^{I\alpha} = U_\alpha^{I\beta} = 0$, its equation of motion has to be $\nabla J_I = 0$. Inspecting the action S_m of (2.1), one verifies that $W^{I\alpha}$ and $U_\alpha^{I\beta}$ couple to J_I similarly to A_M^I , then when these background superfields are non-zero the equation of motion of J_I is

$$\partial J_I = -f_{IJ}^K J_K (\Pi^A A_A^J + d_\alpha W^{J\alpha} + \frac{1}{2} \lambda^\alpha \omega_\beta U_\alpha^{J\beta}). \quad (2.23)$$

The procedure to find the equation of motion of d_α is to first vary the action with respect to Z^M and then multiply the result by the inverse supervielbein E_α^M . Firstly, let us set $\hat{\omega}^\alpha = s^\alpha = 0$ which is equivalent to consider only the minimal action. In this case, one can show that

$$\begin{aligned} \bar{\partial} d_\alpha|_{s=\hat{\omega}=0} &= -\frac{1}{2} \Pi^A \bar{\Pi}^B H_{BA\alpha} + \frac{1}{2} \Pi^a \bar{\Pi}^A T_{A\alpha a} + \frac{1}{2} \bar{\Pi}^a \Pi^A T_{A\alpha a} - \bar{\Pi}^A T_{A\alpha}{}^\beta d_\beta + \bar{\Pi}^A \Omega_{A\alpha}{}^\beta d_\beta \\ &\quad + \lambda^\beta \omega_\gamma \bar{\Pi}^A R_{A\alpha\beta}{}^\gamma + \Pi^A J_I F_{A\alpha}^I + J_I \nabla_\alpha W^{I\beta} d_\beta - \frac{1}{2} \lambda^\beta \omega_\gamma J_I \nabla_\alpha U_\beta^{I\gamma}. \end{aligned}$$

The expression above greatly simplifies when the constraints of the superfields are taken into account. The result is

$$\bar{\partial}d_\alpha|_{s=\hat{\omega}=0} = \bar{\Pi}^A \Omega_{A\alpha}{}^\beta d_\beta + \lambda^\beta \omega_\gamma \bar{\Pi}^A R_{A\alpha\beta}{}^\gamma - \frac{1}{2} \lambda^\beta \omega_\gamma J_I \nabla_\alpha U_\beta^{I\gamma} + \frac{1}{2} J_I U_\alpha^{I\beta} d_\beta.$$

Adding the terms coming from the variation of the non-minimal action and using the definition of $\mathcal{R}_{c\beta ab}$ given in (2.19), the final result is

$$\begin{aligned} \bar{\partial}d_\alpha = & \bar{\Pi}^A \Omega_{A\alpha}{}^\beta d_\beta + \lambda^\beta \omega_\gamma \bar{\Pi}^A R_{A\alpha\beta}{}^\gamma - \frac{1}{2} \lambda^\beta \omega_\gamma J_I \nabla_\alpha U_\beta^{I\gamma} + \frac{1}{2} J_I U_\alpha^{I\beta} d_\beta \\ & - \frac{1}{2} \lambda^\beta J_I U_\beta^{I\gamma} R_{\gamma\alpha}(s\hat{\lambda}) - \bar{\Pi}^A R_{A\alpha}[(sr) + (\hat{\omega}\hat{\lambda})] + \lambda^\beta \bar{\Pi}^A \nabla_\alpha R_{\beta A}(s\hat{\lambda}) + \lambda^\beta \bar{\Pi}^A \nabla_A R_{\alpha\beta}(s\hat{\lambda}) \\ & + \lambda^\beta \bar{\Pi}^A T_{A\alpha}{}^a R_{a\beta}(s\hat{\lambda}) - \frac{1}{4} \lambda^\beta \bar{\Pi}^c \nabla_\alpha \mathcal{R}_{c\beta ab}(s\gamma^{ab}\hat{\lambda}) - \frac{1}{4} \bar{\Pi}^c \mathcal{R}_{c\alpha ab}[(s\gamma^{ab}r) + (\hat{\omega}\gamma^{ab}\hat{\lambda})] \\ & + \frac{1}{2} \lambda^\beta \bar{\Pi}^c \mathcal{R}_{c\beta ad} T_{ab}{}^d (s\gamma^{ab}\hat{\lambda}) + \frac{1}{4} \lambda^\beta \bar{\Pi}^A T_{A\alpha}{}^c \mathcal{R}_{c\beta ab}(s\gamma^{ab}\hat{\lambda}). \end{aligned} \quad (2.24)$$

In what follows, we will also need $\bar{\partial}\Pi^\alpha$ and $\bar{\partial}\bar{\Pi}^a$. One way of computing these equations of motion is using the definitions of Π^A and $\bar{\Pi}^A$ and evaluating sums and differences. Note that

$$\begin{aligned} (\bar{\partial}\Pi^A - \bar{\partial}\bar{\Pi}^A) &= \partial Z^M \bar{\partial} Z^N \partial_{[N} E_{M]}^A \\ &= -\bar{\Pi}^B \Pi^C T_{CB}{}^A - \Pi^B \bar{\Pi}^C \Omega_{CB}{}^A + \bar{\Pi}^B \Pi^C \Omega_{CB}{}^A, \end{aligned} \quad (2.25)$$

where we have used the expression for the torsion (A.1). Using the expression above and $\bar{\Pi}^\alpha = -J_I W^{I\alpha}$, one finds one of the required equations of motion

$$\bar{\partial}\Pi^\alpha = \partial(-J_I W^{I\alpha}) - \bar{\Pi}^a \Pi^b T_{ba}{}^\alpha - \Pi^\beta \bar{\Pi}^C \Omega_{C\beta}{}^\alpha + \bar{\Pi}^\beta \Pi^C \Omega_{C\beta}{}^\alpha. \quad (2.26)$$

In order to find the equation of motion $\bar{\partial}\Pi^a$, it is necessary to use (2.25) with A replaced by a and additionally compute

$$\bar{\partial}\Pi^a + \bar{\partial}\bar{\Pi}^a = 2 \bar{\partial} \partial Z^M E_M{}^a + \partial Z^M \bar{\partial} Z^N \partial_N E_M{}^a + \bar{\partial} Z^M \partial Z^N \partial_N E_M{}^a. \quad (2.27)$$

The combination of terms appearing on the right-hand side of the expression above also appears when varying the action with respect to Z^M and multiplying the result by $E_a{}^M$. In this way, one can compute the left-hand side. Combining (2.25) with (2.27), we have

$$\begin{aligned} \bar{\partial}\Pi_a = & \Pi^b \bar{\Pi}^A \Omega_{Aab} + \bar{\Pi}^b T_{ba}{}^\alpha d_\alpha + \lambda^\alpha \omega_\beta \bar{\Pi}^B R_{aB\alpha}{}^\beta - J_I \nabla_a W^{I\alpha} d_\alpha + \frac{1}{2} \Pi^b J_I U_{ab}^I \\ & + \frac{1}{2} \lambda^\alpha \omega_\beta J_I \nabla_a U_\alpha^{I\beta} + \bar{\Pi}^A R_{Aa}[(sr) + (\hat{\omega}\hat{\lambda})] + \lambda^\alpha \bar{\Pi}^A \nabla_a R_{A\alpha}(s\hat{\lambda}) + \lambda^\alpha \bar{\Pi}^A \nabla_A R_{a\alpha}(s\hat{\lambda}) \\ & - \lambda^\alpha \bar{\Pi}^A T_{Aa}{}^B R_{B\alpha}(s\hat{\lambda}) + \frac{1}{2} \lambda^\alpha J_I U_\alpha^{I\beta} R_{a\beta}(s\hat{\lambda}) - \frac{1}{4} \bar{\Pi}^A \mathfrak{R}_{aAbc}[(s\gamma^{bc}r) + (\hat{\omega}\gamma^{bc}\hat{\lambda})] \\ & + \frac{1}{4} \lambda^\alpha \bar{\Pi}^b \nabla_a \mathcal{R}_{b\alpha cd}(s\gamma^{cd}\hat{\lambda}) - \frac{1}{4} \lambda^\alpha \bar{\Pi}^A \nabla_A \mathcal{R}_{a\alpha bc}(s\gamma^{bc}\hat{\lambda}) + \frac{1}{2} \lambda^\alpha \bar{\Pi}^b \mathcal{R}_{b\alpha cd} T_{ae}{}^d (s\gamma^{ce}\hat{\lambda}) \\ & - \frac{1}{4} \lambda^\alpha \bar{\Pi}^A T_{Aa}{}^b \mathcal{R}_{b\alpha cd}(s\gamma^{cd}\hat{\lambda}) - \frac{1}{2} \lambda^\alpha \bar{\Pi}^A T_{Ab}{}^c \mathcal{R}_{a\alpha cd}(s\gamma^{bd}\hat{\lambda}) + \frac{1}{8} \lambda^\alpha J_I U_\alpha^{I\beta} \mathcal{R}_{a\beta bc}(s\gamma^{bc}\hat{\lambda}), \end{aligned} \quad (2.28)$$

where we have used the definition

$$\mathfrak{R}_{cAab} = R_{cAab} + \nabla_{[A} T_{c]ab} + T_{Ac}{}^B T_{Bab} - T_{Ad[a} T_{b]c}{}^d, \quad (2.29)$$

and (2.19). Note that when A is replaced by α in the above expression, one has $\mathfrak{R}_{c\alpha ab} = \mathcal{R}_{c\alpha ab}$. In what follows, we will also need the equations of motion of the non-minimal variables. Varying the action with respect to $\hat{\omega}^\alpha$, r_α and s_α respectively, we have

$$\begin{aligned} \bar{\partial} \hat{\lambda}_\alpha &= \bar{\Pi}^A \Omega_{A\alpha}{}^\beta \hat{\lambda}_\beta - \frac{1}{4} \bar{\Pi}^A T_{Aab} (\gamma^{ab} \hat{\lambda})_\alpha, & \bar{\partial} s^\alpha &= -s^\beta \bar{\Pi}^A \Omega_{A\beta}{}^\alpha + \frac{1}{4} \bar{\Pi}^A T_{Aab} (s \gamma^{ab})^\alpha, \\ \bar{\partial} r_\alpha &= \bar{\Pi}^A \Omega_{A\alpha}{}^\beta r_\beta - \frac{1}{4} \bar{\Pi}^A T_{Aab} (\gamma^{ab} r)_\alpha + \lambda^\beta \bar{\Pi}^A R_{A\beta} \hat{\lambda}_\alpha + \frac{1}{4} \lambda^\beta \bar{\Pi}^d \mathcal{R}_{d\beta ab} (\gamma^{ab} \hat{\lambda})_\alpha. \end{aligned} \quad (2.30)$$

3 Constructing Ω

In this section, we find a Ω satisfying $\bar{\partial} b = \delta_B \Omega$. Firstly, the case where all the supergravity background superfields are set to zero will be considered. Secondly, we will consider the case where $\hat{\omega}^\alpha = s^\alpha = 0$. Finally, the general case will be considered.

3.1 Super-Yang-Mills background

We begin this subsection by reviewing the construction of Ω in a super-Maxwell background (open superstring) done in [8]. The authors of [8] argued that $\bar{\partial} b$ can be extracted from the OPE between b and the super-Maxwell vertex operator. In the limit where the field-strengths are constant, the super-Maxwell vertex operator reduces to a linear combination of the supersymmetry current and the minimal Lorentz current. As the b ghost is supersymmetric and a Lorentz scalar, they concluded that in that limit Ω is

$$\frac{1}{4} \hat{\lambda}_\beta (\gamma^{ab})_\alpha{}^\beta F_{ab} \frac{\partial b}{\partial r_\alpha}. \quad (3.1)$$

In this subsection, returning to our construction of Ω in a curved heterotic background, the case where all supergravity background superfields are set to zero will be considered. In this case, the b ghost expression given in (2.20) reduces to the flat space one, because $D_\alpha \rightarrow d_\alpha$. In addition, our linearized equations of motion take the same functional form of the equations of motion of an open superstring in a generic super-Maxwell background of [8], apart from the overall $\delta(\sigma)$ term and including J_I and the indices I on the superfields. Note that, due to (2.7), we have $U_\alpha^{I\beta} \rightarrow \frac{1}{2} (\gamma^{ab})_\alpha{}^\beta F_{ab}^I$. Thus, in the case $\partial_a W^{I\alpha} = \partial_a F_{bc}^I = 0$, or in other words considering the field-strengths constants, we can use the results of [8] and conclude that

$$\Omega_0 = \frac{1}{8} (J_I U_{ab}^I) \hat{\lambda}_\beta (\gamma^{ab})_\alpha{}^\beta \frac{\partial b}{\partial r_\alpha}, \quad (3.2)$$

where the subscript 0 in Ω_0 means that Ω reduces to this expression when all the supergravity background superfields are zero, the SYM field-strengths are constant and the equations of motion are linearized. In principle, we can use the results of [8] for the case where the field-strengths are not constant as well, however in [8] the b ghost is not written in terms of RNS-like variables, thus we will derive it again below.

In what follows, we will explicitly check that Ω_0 has the form (3.2) and compute its correction which includes non-linear terms and derivatives of the SYM field-strengths. At order r^0 and setting the supergravity fields to zero, one can compute $\bar{\partial}b$ using the equations of motion and (2.20), the result is the sum of the following three contributions

$$\bar{\partial} \left(-s^\alpha \nabla \hat{\lambda}_\alpha - \frac{1}{4} \Pi^A T_{Aab} (s\gamma^{ab} \hat{\lambda}) - \omega_\alpha \Pi^\alpha \right) |_{\text{SYM}} = \quad (3.3)$$

$$\omega_\alpha \Pi^a J_I \nabla_a W^{I\alpha} - J_K f_{IJ}^K \omega_\alpha W^{I\alpha} d_\beta W^{J\beta} - \frac{1}{2} J_K f_{IJ}^K \omega_\alpha W^{I\alpha} \lambda^\gamma \omega_\beta U_\gamma^{J\beta},$$

$$\bar{\partial} \left(\frac{1}{2(\lambda \hat{\lambda})} (\omega \gamma_a \hat{\lambda}) (\lambda \gamma^a \Pi) \right) |_{\text{SYM}} = \quad (3.4)$$

$$\frac{1}{16(\lambda \hat{\lambda})^2} (\lambda \gamma^{ab} \hat{\lambda}) J_I U_{ab}^I (\omega \gamma_c \hat{\lambda}) (\lambda \gamma^c \Pi) - \frac{1}{16(\lambda \hat{\lambda})} (\omega \gamma_c)^\alpha J_I U_{ab}^I (\gamma^{ab})_\alpha{}^\gamma \hat{\lambda}_\gamma (\lambda \gamma^c \Pi)$$

$$+ \frac{1}{2(\lambda \hat{\lambda})} (\omega \gamma_a \hat{\lambda}) (\lambda \gamma^a)_\alpha \left(J_K f_{IJ}^K W^{I\alpha} d_\beta W^{J\beta} + \frac{1}{2} J_K f_{IJ}^K W^{I\alpha} \lambda^\beta \omega_\gamma U_\beta^{J\gamma} - \Pi^a J_I \nabla_a W^{I\alpha} \right),$$

and finally

$$\bar{\partial}(\Pi^a \bar{I}_a) |_{\text{SYM}, r^0} = -\frac{1}{16(\lambda \hat{\lambda})^2} (\lambda \gamma^{ab} \hat{\lambda}) J_I U_{ab}^I \Pi^c (D \gamma_c \hat{\lambda}) \quad (3.5)$$

$$+ \frac{1}{16(\lambda \hat{\lambda})} \Pi^c (D \gamma_c)^\alpha J_I U_{ab}^I (\gamma^{ab})_\alpha{}^\beta \hat{\lambda}_\beta + \frac{1}{4(\lambda \hat{\lambda})} \Pi^a (\hat{\lambda} \gamma_a)^\alpha \lambda^\beta \omega_\gamma J_I \nabla_\alpha U_\beta^{I\gamma}$$

$$- \frac{1}{4(\lambda \hat{\lambda})} (D \gamma^a \hat{\lambda}) \lambda^\alpha \omega_\beta J_I \nabla_a U_\alpha^{I\beta} + \frac{1}{2(\lambda \hat{\lambda})} (D \gamma^a \hat{\lambda}) J_I \nabla_a W^{I\alpha} d_\alpha.$$

The next step is to compute $\delta_B \Omega_0$ with Ω_0 given in (3.2) at order r^0 in the limit we are considering. Using the BRST transformations of the subsection 2.1.1 and the ones given in (2.17), we have

$$\delta_B \left(\frac{1}{16(\lambda \hat{\lambda})^2} (D \gamma^a \hat{\lambda}) (D \gamma^b \hat{\lambda}) J_I U_{ab}^I \right) |_{\text{SYM}, r^0} = \quad (3.6)$$

$$\frac{1}{8(\lambda \hat{\lambda})^2} \Pi_c (\lambda \gamma^c \gamma^a \hat{\lambda}) (D \gamma^b \hat{\lambda}) J_I U_{ab}^I + \frac{1}{16(\lambda \hat{\lambda})^2} (D \gamma^a \hat{\lambda}) (D \gamma^b \hat{\lambda}) \lambda^\alpha J_I \nabla_\alpha U_{ab}^I,$$

and

$$\delta_B \left(-\frac{1}{128(\lambda \hat{\lambda})^2} \hat{\lambda}_\alpha (\gamma^{ab})_\beta{}^\alpha J_I U_{ab}^I{}^\beta (\gamma_{cef} \hat{\lambda}) (\lambda \gamma^{ef} \omega) \Pi^c \right) |_{\text{SYM}, r^0} = \quad (3.7)$$

$$- \frac{1}{128(\lambda \hat{\lambda})^2} \hat{\lambda}_\alpha (\gamma^{ab})_\beta{}^\alpha J_I U_{ab}^I{}^\beta (\gamma_{cef} \hat{\lambda}) (\lambda \gamma^{ef} D) \Pi^c$$

$$+ \frac{1}{128(\lambda \hat{\lambda})^2} \hat{\lambda}_\alpha (\gamma^{ab})_\beta{}^\alpha J_I U_{ab}^I{}^\beta (\gamma_{cef} \hat{\lambda}) (\lambda \gamma^{ef} \omega) (\lambda \gamma^c \Pi)$$

$$- \frac{1}{128(\lambda \hat{\lambda})^2} \hat{\lambda}_\alpha (\gamma^{ab})_\beta{}^\alpha \lambda^\gamma J_I \nabla_\gamma U_{ab}^I{}^\beta (\gamma_{cef} \hat{\lambda}) (\lambda \gamma^{ef} \omega) \Pi^c.$$

Note that the terms above not having derivatives precisely reproduce the terms without derivatives and linear in the SYM fields of $\bar{\partial}b$ computed previously. To see this, one uses the identity

$$\frac{1}{16}(\gamma^{ab})_{\alpha}{}^{\gamma}(\gamma_{ab})_{\beta}{}^{\delta} = \frac{1}{4}\gamma_{\alpha\beta}^a\gamma_a^{\gamma\delta} - \frac{1}{8}\delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta} - \frac{1}{2}\delta_{\alpha}^{\delta}\delta_{\beta}^{\gamma}, \quad (3.8)$$

the properties of the gamma matrices and the pure spinor condition.

To reproduce all the terms in $\bar{\partial}b$ containing derivatives of the SYM superfields, we need to add terms to Ω_0 . We will make an ansatz for these new terms and then verify that they are the correct ones. The ansatz is

$$\Omega_1 = \frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}\omega_{\alpha} - \frac{1}{4(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^{\beta}(\omega\gamma^b\hat{\lambda}). \quad (3.9)$$

Notice that Ω_1 is gauge invariant under the gauge transformation of ω_{α} of (2.2). This property of Ω_1 becomes manifest when Ω_1 is rewritten using (3.8) as

$$\Omega_1 = -\frac{1}{16(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_{bc})_{\alpha}{}^{\beta}\hat{\lambda}_{\beta}(\lambda\gamma^{bc}\omega) - \frac{1}{8(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}\hat{\lambda}_{\alpha}(\lambda\omega).$$

The BRST transformations of the two terms of Ω_1 are

$$\begin{aligned} \delta_B \left(\frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}\omega_{\alpha} \right) |_{\text{SYM}, r^0} = \\ \frac{1}{2(\lambda\hat{\lambda})}\Pi_b(\lambda\gamma^b\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}\omega_{\alpha} - \frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})\lambda^{\gamma}J_I\nabla_{\gamma}\nabla_a W^{I\alpha}\omega_{\alpha} \\ + \frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}d_{\alpha}, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \delta_B \left(-\frac{1}{4(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^{\beta}(\omega\gamma^b\hat{\lambda}) \right) |_{\text{SYM}, r^0} = \\ -\frac{1}{4(\lambda\hat{\lambda})^2}\Pi_c(\lambda\gamma^c\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^{\beta}(\omega\gamma^b\hat{\lambda}) \\ -\frac{1}{4(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^{\beta}(D\gamma^b\hat{\lambda}) \\ +\frac{1}{4(\lambda\hat{\lambda})^2}(D\gamma^a\hat{\lambda})\lambda^{\gamma}J_I\nabla_{\gamma}\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^{\beta}(\omega\gamma^b\hat{\lambda}). \end{aligned} \quad (3.11)$$

To see that in fact all the derivative terms in $\bar{\partial}b$ are reproduced requires several manipulations, the use of the superfields constraints, the Bianchi identities and the following property of the covariant derivatives

$$[\nabla_A, \nabla_B]_{\pm} W^{I\alpha} = -T_{AB}{}^C \nabla_C W^{I\alpha} + R_{AB\beta}{}^{\alpha} W^{I\beta} - f_{JK}^I F_{AB}^K W^{J\alpha}, \quad (3.12)$$

where \pm means commutator or anticommutator depending on the grading of A and B . Firstly, note that the last term of (3.10) is equal to the last term of (3.5). Secondly, using

the constraint $\frac{1}{2}U_\beta^{I\alpha} = \nabla_\beta W^{I\alpha}$ and (3.12) with the appropriate indices, we can rewrite the third term on the right-hand side of (3.5) as

$$\begin{aligned} \frac{1}{4(\lambda\hat{\lambda})}\Pi^a(\hat{\lambda}\gamma_a)^\alpha\lambda^\beta\omega_\gamma J_I\nabla_\alpha U_\beta^{I\gamma} &= \frac{1}{2(\lambda\hat{\lambda})}\Pi^a(\hat{\lambda}\gamma_a)^\alpha\lambda^\beta\omega_\gamma J_I\nabla_\alpha\nabla_\beta W^{I\gamma} = \\ &-\frac{1}{4(\lambda\hat{\lambda})}\Pi^a(\hat{\lambda}\gamma_a)^\alpha\lambda^\beta\omega_\gamma J_I\nabla_\beta U_\alpha^{I\gamma} - \omega_\alpha\Pi^a J_I\nabla_a W^{I\alpha} + \frac{1}{2(\lambda\hat{\lambda})}\Pi_b(\lambda\gamma^b\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}\omega_\alpha. \end{aligned} \quad (3.13)$$

It is immediately to see that the second term in the second line of the expression above cancels the first term on the right-hand side of (3.3) and the third term is equal to the first term on the right-hand side of (3.10). Similarly, one has that the fourth term in (3.5) takes the form

$$\begin{aligned} -\frac{1}{4(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})\lambda^\alpha\omega_\beta J_I\nabla_a U_\alpha^{I\beta} &= -\frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})\lambda^\alpha\omega_\beta J_I\nabla_a\nabla_\alpha W^{I\beta} = \\ &-\frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})\lambda^\gamma J_I\nabla_\gamma\nabla_a W^{I\alpha}\omega_\alpha + \frac{1}{2(\lambda\hat{\lambda})}(D\gamma^a\hat{\lambda})\lambda^\gamma J_I f^I_{JK}F_{a\gamma}^K W^{J\alpha}\omega_\alpha. \end{aligned} \quad (3.14)$$

Note that the first term in the second line of the expression above is equal to the second term on the right-hand side of (3.10). To finish the proof that all the terms with derivatives in $\bar{\partial}b$ are reproduced, we need to manipulate some of the terms coming from the BRST variations as well. Using the identity (3.8), the pure spinor condition, $F_{\alpha\beta} = 0$ and (3.12), one can show that the last term of (3.7) takes the form

$$\begin{aligned} -\frac{1}{128(\lambda\hat{\lambda})^2}\hat{\lambda}_\alpha(\gamma^{ab})_\beta{}^\alpha\lambda^\gamma J_I\nabla_\gamma U_{ab}^{I\beta}(\gamma_{cef}\hat{\lambda})(\lambda\gamma^{ef}\omega)\Pi^c = \\ -\frac{1}{4(\lambda\hat{\lambda})}\Pi^a(\hat{\lambda}\gamma_a)^\alpha\lambda^\beta\omega_\gamma J_I\nabla_\beta U_\alpha^{I\gamma} - \frac{1}{8(\lambda\hat{\lambda})^2}\Pi^a(\lambda\gamma^b\gamma_a\hat{\lambda})(\omega\gamma^c\hat{\lambda})\lambda^\alpha J_I\nabla_\alpha U_{cb}^I. \end{aligned} \quad (3.15)$$

One can verify that the first term on the right-hand side of the expression above is equal to the first term in the second line of (3.13). In addition, using the following identities valid in the limit we are considering (the second one follows from $\frac{1}{2}U_\alpha^{I\beta} = \nabla_\alpha W^{I\beta}$ and the pure spinor condition)

$$(\gamma^a)_{\alpha\beta}\nabla_a W^{I\gamma} = -(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha)W^{I\gamma}, \quad \lambda^\alpha(\lambda\gamma_a)_\beta(\nabla_\alpha W^{I\beta}) = 0, \quad (3.16)$$

one can show that the first term on the right-hand side of (3.11) is

$$\begin{aligned} -\frac{1}{4(\lambda\hat{\lambda})^2}\Pi_c(\lambda\gamma^c\gamma^a\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^\beta(\omega\gamma^b\hat{\lambda}) = \\ \frac{1}{4(\lambda\hat{\lambda})^2}\Pi_c(\lambda\gamma^a\gamma^c\hat{\lambda})J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^\beta(\omega\gamma^b\hat{\lambda}) - \frac{1}{2(\lambda\hat{\lambda})}\Pi^a J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^\beta(\omega\gamma^b\hat{\lambda}) = \\ \frac{1}{8(\lambda\hat{\lambda})^2}\Pi^a(\lambda\gamma^b\gamma_a\hat{\lambda})(\omega\gamma^c\hat{\lambda})\lambda^\alpha J_I\nabla_\alpha U_{cb}^I - \frac{1}{2(\lambda\hat{\lambda})}\Pi^a J_I\nabla_a W^{I\alpha}(\gamma_b)_{\alpha\beta}\lambda^\beta(\omega\gamma^b\hat{\lambda}). \end{aligned}$$

Note that the last term above is equal to the last term in (3.4) and the first term in the last line above cancels the last term of (3.15). In addition, using the Bianchi identity (A.3)

in the limit where the supergravity superfields are zero, it is possible to show that the last term in (3.6) cancels the second term on the right-hand side of (3.11). Finally, one can show that the last term in (3.11) is equal to

$$\begin{aligned} & \frac{1}{4(\lambda\hat{\lambda})^2} (D\gamma^a \hat{\lambda}) \lambda^\gamma J_I \nabla_\gamma \nabla_a W^{I\alpha} (\gamma_b)_{\alpha\beta} \lambda^\beta (\omega \gamma^b \hat{\lambda}) = \\ & - \frac{1}{4(\lambda\hat{\lambda})^2} (D\gamma^a \hat{\lambda}) \lambda^\gamma J_I f^I_{JK} F_{\gamma a}^K W^{J\alpha} (\gamma_b)_{\alpha\beta} \lambda^\beta (\omega \gamma^b \hat{\lambda}). \end{aligned} \quad (3.17)$$

This concludes the proof that all terms linear in the SYM superfields containing derivatives or not in $\bar{\partial}b$ are reproduced by $\delta_B(\Omega_0 + \Omega_1)$ at order r^0 as only non-linear terms are left. The non-linear terms in $\bar{\partial}b$ come from (3.3), (3.4) and (3.14), they are

$$\begin{aligned} & -J_K f_{IJ}^K \omega_\alpha W^{I\alpha} d_\beta W^{J\beta} - \frac{1}{2} J_K f_{IJ}^K \omega_\alpha W^{I\alpha} \lambda^\gamma \omega_\beta U_\gamma^{J\beta} \\ & + \frac{1}{2(\lambda\hat{\lambda})} (\omega \gamma_a \hat{\lambda}) (\lambda \gamma^a)_\alpha \left(J_K f_{IJ}^K W^{I\alpha} d_\beta W^{J\beta} + \frac{1}{2} J_K f_{IJ}^K W^{I\alpha} \lambda^\beta \omega_\gamma U_\beta^{J\gamma} \right) \\ & + \frac{1}{2(\lambda\hat{\lambda})} (D\gamma^a \hat{\lambda}) \lambda^\gamma J_I f^I_{JK} F_{a\gamma}^K W^{J\alpha} \omega_\alpha. \end{aligned}$$

The only non-linear term from the previous BRST variations is the last one in (3.17). It is not difficult to see, using the BRST transformations of both the subsection 2.1.1 and (2.17) and the constraints (2.5), that $\delta_B \Omega_2$ with

$$\begin{aligned} \Omega_2 = & \frac{1}{2} J_K f_{IJ}^K \omega_\alpha W^{I\alpha} \omega_\beta W^{J\beta} - \frac{1}{2(\lambda\hat{\lambda})} J_K f_{IJ}^K (\omega \gamma^a \hat{\lambda}) (\lambda \gamma_a)_\alpha W^{I\alpha} \omega_\beta W^{J\beta} \\ & + \frac{1}{8(\lambda\hat{\lambda})^2} J_K f_{IJ}^K W^{I\alpha} (\gamma_a)_{\alpha\beta} \lambda^\beta (\omega \gamma^a \hat{\lambda}) W^{J\gamma} (\gamma_b)_{\gamma\delta} \lambda^\delta (\omega \gamma^b \hat{\lambda}), \end{aligned} \quad (3.18)$$

reproduces all the non-linear terms in $\bar{\partial}b$ and cancels the non-linear term coming from the previous BRST variations. Note that Ω_2 can be rewritten as manifestly gauge invariant as

$$\begin{aligned} \Omega_2 = & \frac{1}{2(\lambda\hat{\lambda})^2} J_K f_{IJ}^K \left[\frac{1}{8} W^{I\alpha} (\gamma_{ab})_\alpha{}^\beta \hat{\lambda}_\beta (\lambda \gamma^{ab} \omega) + \frac{1}{4} W^{I\alpha} \hat{\lambda}_\alpha (\lambda \omega) \right] \\ & \times \left[\frac{1}{8} W^{J\gamma} (\gamma_{cd})_\gamma{}^\delta \hat{\lambda}_\delta (\lambda \gamma^{cd} \omega) + \frac{1}{4} W^{J\gamma} \hat{\lambda}_\gamma (\lambda \omega) \right], \end{aligned}$$

where we have used (3.8).

This concludes the construction of Ω at order r^0 in a super-Yang-Mills heterotic background when all the background supergravity superfields are set to zero. The next step is to construct Ω at order r in the same limit. To achieve this, one first compute $\bar{\partial}b$ at order r using the equations of motion. It is possible to show that all the linear terms without derivatives are reproduced by $\delta_B \Omega_0$ where Ω_0 is given in (3.2). One can show that the terms from $\delta_B \Omega_0$ containing derivatives together with both the terms of order r of $\delta_B \Omega_1$ with Ω_1 given in (3.9) and $\delta_B \Omega_3$ with

$$\Omega_3 = -\frac{1}{16(\lambda\hat{\lambda})^2} [J_I \nabla_c W^{I\alpha} \omega_\alpha - \frac{1}{2(\lambda\hat{\lambda})} J_I \nabla_c W^{I\alpha} (\gamma_d)_{\alpha\beta} \lambda^\beta (\omega \gamma^d \hat{\lambda})] (r \gamma^c{}_{ab} \hat{\lambda}) (\lambda \gamma^{ab} \omega),$$

reproduce all the linear terms of order r in $\bar{\partial}b$. The procedure to prove this is similar to the one described at order r^0 and the details will be omitted. In addition, all the non-linear terms are canceled or reproduced by $\delta_B\Omega_2$. At order r^2 and r^3 , one follows the same steps described for lower orders in r . It is possible to show that all terms in $\bar{\partial}b$ at this order are reproduced by $\delta_B(\Omega_0 + \Omega_3)$, however the details will be again omitted as the procedure is similar to the case r^0 . This concludes the construction of Ω in a super-Yang-Mills background.

3.2 The case where $s^\alpha = \hat{\omega}^\alpha = 0$

In this section, the case where both the supergravity and super-Yang-Mills background superfields are non-vanishing will be considered. However, a simplifying assumption will be made, we are going to consider $s^\alpha = \hat{\omega}^\alpha = 0$. In order to construct Ω , we compute $\bar{\partial}b$ using the equations of motion. Observing the result, one verify after a few manipulations that at any order in r the terms linear in $\bar{\Pi}^A T_{Aab}$ can be combined with the terms linear in $J_I U_{ab}^I$ as a function of the combination $\bar{\Pi}^A T_{Aab} + \frac{1}{2} J_I U_{ab}^I$. So, we make the ansatz that Ω_0 of (3.2) has to be modified to

$$(\Omega_0)_{\text{new}} = \frac{1}{4} \left(\bar{\Pi}^A T_{Aab} + \frac{1}{2} J_I U_{ab}^I \right) \hat{\lambda}_\beta (\gamma^{ab})_\alpha{}^\beta \frac{\partial b}{\partial r_\alpha}. \quad (3.19)$$

In addition, further analyzing the result of $\bar{\partial}b$, we add new terms to Ω_1 and Ω_3 , which are

$$\Omega_4 = \bar{\Pi}^b T_{ab}{}^\alpha \omega_\alpha \bar{F}^a - \frac{1}{2(\lambda\hat{\lambda})} \bar{\Pi}^b T_{ab}{}^\alpha (\gamma_c)_{\alpha\beta} \lambda^\beta (\omega \gamma^c \hat{\lambda}) \bar{F}^a, \quad (3.20)$$

where \bar{F}^a are the generalized RNS-like variables defined in (2.21). Using the BRST transformations, one can show that indeed all the terms in $\bar{\partial}b$ with $s^\alpha = \hat{\omega}^\alpha = 0$ are reproduced. For the terms containing the super-Yang-Mills fields, one follows the procedure describe in the previous subsection, but this time keeping the terms with supergravity background fields. To show that the remaining terms in $\bar{\partial}b$ are reproduced, one uses (2.9) and the following identities. The Bianchi identity (A.2) with the indices $[\beta ab]$ and the constraints (2.5) imply

$$\nabla_\beta T_{ab}{}^\alpha = R_{ab\beta}{}^\alpha - T_{\beta a}{}^c T_{cb}{}^\alpha - T_{b\beta}{}^c T_{ca}{}^\alpha.$$

The same Bianchi identity (A.2), but with the indices $[\alpha\beta a]$ implies

$$(\gamma^b)_{\alpha\beta} T_{ba}{}^\gamma = R_{a\alpha\beta}{}^\gamma - R_{\beta a\alpha}{}^\gamma.$$

Finally, one can show that

$$R_{c\alpha ab} = \nabla_c T_{\alpha ab} + T_{ac}{}^\beta (\gamma_b)_{\beta\alpha} + T_{cb}{}^\beta (\gamma_a)_{\beta\alpha}.$$

In order to prove the identity above, we have used that

$$R_{c\alpha ab} = \frac{1}{4} (R_{[c\alpha a]b} - R_{[cab]a} + R_{[\alpha ab]c}), \quad (3.21)$$

the Bianchi Identity (A.2) three times, the Bianchi Identity (A.5) and $T_{abc} = -H_{abc}$ as proved in (2.8).

3.3 The general case

In this subsection, we will consider the general case. All the terms in $\bar{\partial}b$ not containing either s^α or $\hat{\omega}^\alpha$ have already been written as BRST exact in the previous two subsections. Thus, the remaining terms at order r^0 in $\bar{\partial}b$ are

$$\bar{\partial} \left(-s^\alpha \nabla \hat{\lambda}_\alpha - \frac{1}{4} \Pi^A T_{Aab} (s\gamma^{ab} \hat{\lambda}) - \omega_\alpha \Pi^\alpha \right) |_{s, \hat{\omega}} = \quad (3.22)$$

$$\Pi^d \bar{\Pi}^A R_{Ad} (s\hat{\lambda}) - \frac{1}{4} \bar{\Pi}^A \Pi^d \Re_{dAab} (s\gamma^{ab} \hat{\lambda}),$$

$$\bar{\partial} \left(\frac{1}{2(\lambda\hat{\lambda})} (\omega\gamma_a \hat{\lambda}) (\lambda\gamma^a \Pi) \right) |_{s, \hat{\omega}} = \quad (3.23)$$

$$\frac{1}{2(\lambda\hat{\lambda})} (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^A R_{A\alpha} (s\hat{\lambda}) (\lambda\gamma^a \Pi) + \frac{1}{8(\lambda\hat{\lambda})} (\hat{\lambda}\gamma_d)^\alpha \bar{\Pi}^c \mathcal{R}_{c\alpha ab} (s\gamma^{ab} \hat{\lambda}) (\lambda\gamma^d \Pi),$$

and finally

$$\bar{\partial}(\Pi^a \bar{F}_a) |_{s, \hat{\omega}, r^0} = -\frac{1}{2(\lambda\hat{\lambda})} (D\gamma_a \hat{\lambda}) (\bar{\partial}\Pi^a) |_{s, \hat{\omega}} - \frac{1}{2(\lambda\hat{\lambda})} \Pi^a (\hat{\lambda}\gamma_a)^\alpha (\bar{\partial}d_\alpha) |_{s, \hat{\omega}} \quad (3.24)$$

$$+ \frac{3(\lambda\Omega)}{2(\lambda\hat{\lambda})} \Pi^a (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^A R_{A\alpha} (s\hat{\lambda}) - \frac{1}{8(\lambda\hat{\lambda})} \Pi^c (\hat{\lambda}\gamma_c)^\alpha \lambda^\beta T_{\beta ab} (\gamma^{ab})_\alpha{}^\gamma \bar{\Pi}^A R_{A\gamma} (s\hat{\lambda})$$

$$+ \frac{3(\lambda\Omega)}{8(\lambda\hat{\lambda})} \Pi^d (\hat{\lambda}\gamma_d)^\alpha \bar{\Pi}^c \mathcal{R}_{c\alpha ab} (s\gamma^{ab} \hat{\lambda}) - \frac{1}{32(\lambda\hat{\lambda})} \Pi^c (\hat{\lambda}\gamma_c)^\alpha \lambda^\beta T_{\beta ab} (\gamma^{ab})_\alpha{}^\gamma \bar{\Pi}^d \mathcal{R}_{d\gamma ef} (s\gamma^{ef} \hat{\lambda}),$$

where $\bar{\partial}d_\alpha |_{s, \hat{\omega}}$ and $\bar{\partial}\Pi^a |_{s, \hat{\omega}}$ can be read from (2.24) and (2.28) respectively. Let us consider first the terms involving the curvatures with two indices. One can show that the terms above from $\bar{\partial}b$ can be written as $\delta_B \Omega_{0s} |_{r^0}$ with

$$\Omega_{0s} = -\frac{1}{2(\lambda\hat{\lambda})} \Pi^a (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^A R_{A\alpha} (s\hat{\lambda}) + \frac{1}{2(\lambda\hat{\lambda})} (D\gamma^a \hat{\lambda}) \bar{\Pi}^A R_{Aa} (s\hat{\lambda}). \quad (3.25)$$

The proof is straightforward, it follows from the identities

$$\nabla_{[A} R_{BC]} + T_{[AB}{}^F R_{|F|C]} = 0, \quad \bar{\nabla} \lambda^\alpha = -\frac{1}{2} \lambda^\beta J_I U_\beta^{I\alpha}.$$

One can compute the terms in $\bar{\partial}b$ of higher orders in r using the equations of motion. In this case, one has to add to Ω_{0s} terms that depend on r . The additional terms are given in the next section and the proof that they are the correct ones only uses the two identities above.

Removing the terms with curvatures with two indices from (3.22), (3.23) and (3.24), we are left with terms with curvatures with four indices and torsions. These remaining terms can also be written as BRST exact. They are equal to $\delta_B \Omega_{1s} |_{r^0}$, where

$$\Omega_{1s} = -\frac{1}{8(\lambda\hat{\lambda})} \Pi^c (\hat{\lambda}\gamma_c)^\alpha \bar{\Pi}^d \mathcal{R}_{d\alpha ab} (s\gamma^{ab} \hat{\lambda}) - \frac{1}{8(\lambda\hat{\lambda})} (D\gamma^c \hat{\lambda}) \bar{\Pi}^D \Re_{cDab} (s\gamma^{ab} \hat{\lambda}).$$

To prove it, we have used the following two non-trivial identities

$$\begin{aligned}
 (\gamma^c)_{\alpha\beta} \mathfrak{R}_{cAab} &= \nabla_\alpha \mathcal{R}_{A\beta ab} + \nabla_\beta \mathcal{R}_{A\alpha ab} \\
 &\quad - T_{A\alpha}{}^c \mathcal{R}_{c\beta ab} - T_{A\beta}{}^c \mathcal{R}_{c\alpha ab} - \mathcal{R}_{A\alpha c[a} T_{b]\beta}{}^c - \mathcal{R}_{A\beta c[a} T_{b]\alpha}{}^c,
 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
 \nabla_{[c} \mathcal{R}_{A]\alpha ab} + T_{cA}{}^B \mathcal{R}_{B\alpha ab} + \mathcal{R}_{A\alpha d[a} T_{b]c}{}^d - (-)^A \mathcal{R}_{\alpha d[a} T_{b]A}{}^d \\
 = T_{A\alpha}{}^d \mathfrak{R}_{cdab} + (-)^A T_{c\alpha}{}^d \mathfrak{R}_{dAab} - (-)^A \nabla_\alpha \mathfrak{R}_{cAab} - \mathfrak{R}_{cAd[a} T_{b]\alpha}{}^d.
 \end{aligned} \tag{3.27}$$

These identities can be checked after tedious calculations by substituting the definitions of \mathcal{R} and \mathfrak{R} given in (2.19) and (2.29), using the Bianchi Identities (A.2) and (A.4) several times and the following relation among covariant derivatives

$$\begin{aligned}
 [\nabla_A, \nabla_B]_{\pm} T_{CD}{}^E &= -T_{AB}{}^F \nabla_F T_{CD}{}^E \\
 &\quad - R_{ABC}{}^F T_{FD}{}^E - (-)^{C(D+F)} R_{ABD}{}^F T_{CF}{}^E + (-)^{(A+B)(F+C+D)} T_{CD}{}^F R_{ABF}{}^E,
 \end{aligned}$$

where \pm means commutator or anticommutator depending on A and B .

This completes the construction of Ω at order r^0 . Using the equations of motion to evaluate $\bar{\partial}b$, it is easy to find the terms with s^α and $\hat{\omega}^\alpha$ which depends on \mathcal{R} and \mathfrak{R} and are higher order in r . These terms are again BRST exact if one modifies Ω_{1s} by terms depending on r . These additional terms of Ω_{1s} are given in the next section and the proof that they are the correct ones is straightforward and follows from the equalities (3.26) and (3.27).

4 The complete Ω

In this section, the complete expression for Ω is presented. The Ω is a solution of the equation $\bar{\partial}b = \delta_B \Omega$ and it is defined up to a BRST exact term. Our Ω is

$$\begin{aligned}
 \Omega &= \frac{1}{2} J_K f_{IJ}^K \omega_\alpha W^{I\alpha} \omega_\beta W^{J\beta} - \frac{1}{2(\lambda\hat{\lambda})} J_K f_{IJ}^K W^{I\alpha} (\gamma_a)_{\alpha\beta} \lambda^\beta (\omega\gamma^a \hat{\lambda}) \omega_\gamma W^{J\gamma} \\
 &\quad + \frac{1}{8(\lambda\hat{\lambda})^2} J_K f_{IJ}^K W^{I\alpha} (\gamma_a)_{\alpha\beta} \lambda^\beta (\omega\gamma^a \hat{\lambda}) W^{J\gamma} (\gamma_b)_{\gamma\delta} \lambda^\delta (\omega\gamma^b \hat{\lambda}) \\
 &\quad + \frac{1}{4} (\bar{\Pi}^A T_{Aab} + \frac{1}{2} J_I U_{ab}^I) \hat{\lambda}_\beta (\gamma^{ab})_\alpha{}^\beta \frac{\partial b}{\partial r_\alpha} \\
 &\quad + (J_I \nabla_a W^{I\alpha} + \bar{\Pi}^b T_{ab}{}^\alpha) \omega_\alpha \bar{F}^a - \frac{1}{2(\lambda\hat{\lambda})} (J_I \nabla_a W^{I\alpha} + \bar{\Pi}^b T_{ab}{}^\alpha) (\gamma_c)_{\alpha\beta} \lambda^\beta (\omega\gamma^c \hat{\lambda}) \bar{F}^a \\
 &\quad - \frac{1}{2(\lambda\hat{\lambda})} \Pi^a (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^A R_{A\alpha}(s\hat{\lambda}) + \bar{\Pi}^A R_{Aa}(s\hat{\lambda}) \bar{F}^a - \frac{1}{4(\lambda\hat{\lambda})^2} (\lambda\gamma^{ab} r) (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^A R_{A\alpha}(s\hat{\lambda}) \bar{F}_b \\
 &\quad - \frac{1}{8(\lambda\hat{\lambda})} \Pi^c (\hat{\lambda}\gamma_c)^\alpha \bar{\Pi}^d \mathcal{R}_{d\alpha ab} (s\gamma^{ab} \hat{\lambda}) - \frac{1}{4} \bar{\Pi}^D \mathfrak{R}_{cDab} (s\gamma^{ab} \hat{\lambda}) \bar{F}^c \\
 &\quad - \frac{1}{16(\lambda\hat{\lambda})^2} (\lambda\gamma^{ab} r) (\hat{\lambda}\gamma_a)^\alpha \bar{\Pi}^D \mathcal{R}_{D\alpha cd} (s\gamma^{cd} \hat{\lambda}) \bar{F}_b.
 \end{aligned}$$

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A Notations and conventions

In this paper, we mainly follow the conventions of Wess and Bagger [14] and of Chandia and Vallilo [13]. The curved superspace indices are denoted by M and take the values $M = (m, \mu)$ with $m = 0, \dots, 9$ and $\mu = 1, \dots, 16$. The tangent superspace indices are denoted by $A = (a, \alpha)$, with $a = 0, \dots, 9$ and $\alpha = 1, \dots, 16$. Being Z^M the $\mathcal{N} = 1$ $D = 10$ curved superspace coordinates, one has

$$Z^M Z^N = (-)^{MN} Z^N Z^M.$$

In order to define differential forms, one introduces the exterior product, which obeys

$$dZ^M \wedge dZ^N = -(-)^{MN} dZ^N \wedge dZ^M.$$

A p -form Θ is defined by

$$\Theta = \frac{1}{p!} dZ^{M_1} \wedge dZ^{M_2} \wedge \dots \wedge dZ^{M_p} \Theta_{M_p \dots M_1}.$$

The exterior derivative d is an operator which maps p -forms into $p+1$ -forms, its action on Θ is

$$d\Theta = \frac{1}{p!} dZ^{M_1} \wedge dZ^{M_2} \wedge \dots \wedge dZ^{M_p} \wedge dZ^N \frac{\partial}{\partial Z^N} \Theta_{M_p \dots M_1}.$$

An important property of the exterior derivative that follows from the above definitions is

$$d(\Theta \Sigma) = \Theta(d\Sigma) + (-)^q (d\Theta) \Sigma,$$

where Σ is a q -form.

A.1 Connections and covariant derivatives

Three different connections appear in this paper: the gauge connection, the Lorentz connection and the scaling connection. The one-form gauge connection taking values in a Lie Algebra is

$$A = dZ^M A_M^I T_I,$$

where T_I are the Lie Algebra generators. The one-form Lorentz connection and scaling connection are denoted respectively by

$$\Omega_a{}^b = dZ^M \Omega_{Ma}{}^b, \quad \hat{\Omega}_s = dZ^M \Omega_M,$$

with $\Omega_{Mab} = -\Omega_{Mba}$. These two connections appear in the decomposition of $\Omega_{M\alpha}{}^\beta$ as

$$\Omega_{M\alpha}{}^\beta = \Omega_M \delta_\alpha^\beta + \frac{1}{4} \Omega_{Mab} (\gamma^{ab})_\alpha{}^\beta.$$

Using the one-form connections given above, one can define covariant derivatives of forms. Consider a p -form $\Sigma_A{}^B = \Sigma_A^{IB} T_I$ taking values in a Lie Algebra. The covariant derivative of $\Sigma_A{}^B$ is defined by

$$\nabla \Sigma_A{}^B = d\Sigma_A{}^B - \Omega_A{}^C \wedge \Sigma_C{}^B + \Sigma_A{}^C \wedge \Omega_C{}^B - \Sigma_A{}^B \wedge A + (-)^p A \wedge \Sigma_A{}^B.$$

Using the previous definitions and

$$\nabla \Sigma_A{}^B = \frac{1}{p!} dZ^{M_1} \wedge dZ^{M_2} \wedge \dots \wedge dZ^{M_p} \wedge dZ^N \nabla_N (\Sigma_{M_p \dots M_1}^I)_A{}^B T_I,$$

it is possible to deduce an expression for the covariant derivative of the components of the form $\Sigma_A{}^B$.

A.2 Supervielbein, torsions and curvatures

The one-form supervielbein is defined by

$$E^A = dZ^M E_M{}^A.$$

The inverse supervielbein matrix is defined implicitly by the relations

$$E_M{}^A E_A{}^N = \delta_M^N, \quad E_A{}^M E_M{}^B = \delta_A^B.$$

Using the definitions of the one-form supervielbein, the one-form connections and the covariant derivative, one defines the two-form torsion T^A as

$$T^A = \nabla E^A = dE^A + E^B \wedge \Omega_B{}^A = \frac{1}{2} dZ^M \wedge dZ^N T_{NM}{}^A = \frac{1}{2} E^C \wedge E^B T_{BC}{}^A,$$

or in components,

$$T_{NM}{}^A = \partial_{[N} E_{M]}{}^A + (-)^{N(B+M)} E_M{}^B \Omega_{NB}{}^A - (-)^{MB} E_N{}^B \Omega_{MB}{}^A, \quad (\text{A.1})$$

where the brackets means antisymmetrization of the indices without any additional numerical factor, i.e. $\partial_{[N} E_{M]}{}^A = \partial_N E_M{}^A - (-)^{MN} \partial_M E_N{}^A$. The curvature two-form $R_A{}^B$ is given by

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B = \frac{1}{2} dZ^M \wedge dZ^N R_{NM}{}^B = \frac{1}{2} E^C \wedge E^D R_{DC}{}^B,$$

or in components

$$R_{NMA}{}^B = \partial_{[N} \Omega_{M]A}{}^B + (-)^{N(M+A+C)} \Omega_{MA}{}^C \Omega_{NC}{}^B - (-)^{M(A+C)} \Omega_{NA}{}^C \Omega_{MC}{}^B.$$

Note that due to the fact that $\Omega_{M\alpha}{}^\beta$ has the decomposition (2.3), it is possible to show that

$$R_{NM\alpha}{}^\beta = R_{NM} \delta_\alpha^\beta + \frac{1}{4} R_{NMab} (\gamma^{ab})_\alpha{}^\beta,$$

where the curvature with two indices is constructed from the scaling connection.

A.3 The super-Yang-Mills field-strength and the three-form field-strength

The two-form super-Yang-Mills field-strength is constructed from the gauge connection as

$$F = F^I T_I = dA - A \wedge A = \frac{1}{2} dZ^M \wedge dZ^N F_{NM} = \frac{1}{2} E^A \wedge E^B F_{BA},$$

or in components,

$$F_{NM}^I = \partial_{[N} A_{M]}^I - (-)^{NM} A_M^J A_N^K f_{JK}^I,$$

where f_{JK}^I are the structure constants of a Lie Algebra, $[T_J, T_K] = f_{JK}^I T_I$. The three-form field-strength H is written in terms of the two-form superfield potential B as

$$H = dB = \frac{1}{6} dZ^M \wedge dZ^N \wedge dZ^P H_{PNM} = \frac{1}{6} E^C \wedge E^B \wedge E^A H_{ABC},$$

or in components

$$H_{PNM} = \frac{1}{2} \partial_{[P} B_{NM]},$$

where again the brackets mean antisymmetrization of the indices with no additional numerical factors.

A.4 Bianchi identities

In this subsection, we collect all the Bianchi identities. Using the definitions of the previous subsections of this appendix, we have

$$T^D = \nabla E^D, \quad \nabla T^D = E^B \wedge R_B^D \Rightarrow \nabla_{[A} T_{BC]}^D - R_{[ABC]}^D + T_{[AB}^F T_{F|C]}^D = 0, \quad (\text{A.2})$$

$$\nabla F = 0 \Rightarrow \nabla_{[A} F_{BC]}^I + T_{[AB}^D F_{D|C]}^I = 0, \quad (\text{A.3})$$

$$\nabla R_D^E = 0 \Rightarrow \nabla_{[A} R_{BC]D}^E + T_{[AB}^F R_{F|C]D}^E = 0, \quad (\text{A.4})$$

$$dH = 0 \Rightarrow \nabla_{[A} H_{BCD]} + \frac{3}{2} T_{[AB}^E H_{E|CD]} = 0, \quad (\text{A.5})$$

where the indices inside the brackets but outside the double bars $||$ are antisymmetrized.

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